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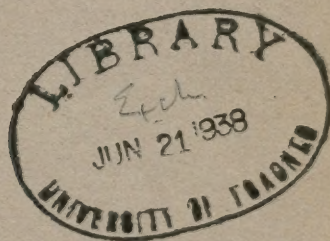
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1916

The Texas Mathematics Teachers' Bulletin

(Vol. 1, No. 2, January 25, 1916)



Published by the University six times a month and entered as
second-class matter at the postoffice at
AUSTIN, TEXAS

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If you wish to receive this bulletin regularly tear out this insert, fill in the necessary information and mail to J. W. Calhoun or C. D. Rice, University Station, Austin, Texas.

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1916: No. 6

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The Texas Mathematics Teachers' Bulletin

(Vol. 1, No. 2, January 25, 1916)

Edited by

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Adjunct Professor of Pure Mathematics,

and

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This Bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston.

Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

Mirabeau B. Lamar.

MATHEMATICS TEACHERS' WEEK

June 14-17, 1916

There will be held at the University of Texas June 14-17, 1916, a four-day conference on the teaching of high school mathematics. The conference will be held under the auspices of the University Summer School and will be conducted by members of the faculty of Pure and Applied Mathematics.

It will be open free of cost to all teachers of mathematics. It is earnestly hoped that many of the teachers of the state will attend. The object of the conference is to afford an opportunity for the co-operative study of the problems connected with the teaching of secondary mathematics, and a mutually helpful interchange of ideas.

Teachers who are interested are invited to write to Prof. M. B. Porter who will be glad to furnish information or receive any suggestions looking to making the week more profitable to all who attend.

The conference will be held in Room 157 of the Main Building and will be conducted in accordance with the following schedule:

Arithmetic and Algebra.

Lecture, 8 to 9:30.

Conference, 9:30 to 10:30.

Professor Benedict.

Plane Geometry.

Lecture, 11 to 12.

Conference, 12 to 1.

Professor Calhoun.

Solid Geometry.

Lecture, 2:30 to 3:25 p. m.

Conference, 3:25 to 4:20 p. m.

Professor Porter.

Trigonometry.

Lecture, 4:20 to 5:15 p. m.

Conference, 5:15 to 6 p. m.

Professor Rice.

The lectures will at least touch upon all the important topics in the high school course and the conferences will afford an opportunity for all present to discuss profitably any questions and difficulties that may arise.

ANNUAL MEETING OF THE MATHEMATICS SECTION
OF THE TEXAS STATE TEACHERS'
ASSOCIATION

The Mathematics Section of the Texas State Teachers' Association met in the W. O. W. hall at Corpus Christi on the morning of November 26, 1915. The chairman, Prof. W. L. Abbott of Houston, presided. There were about seventy-five members present. The following program was carried out:

1. Does the Student Get a Fair Return for the Time and Effort Spent on Mathematics?—T. H. Rogers, High School, Houston.
2. Elementary Mathematics Preparatory to the Mathematics of Investment and Insurance.—Dr. E. L. Dodd, University of Texas.
3. The Symbolism of Algebra and of Geometry.—Miss Pearl Crawford, High School, Corpus Christi.
4. Incommensurable Ratios and Limits in High School Geometry.—Prof. W. C. Vernon, S. W. T. N., San Marcos.
5. Proofs, by Approximation Methods, of Certain Theorems of Geometry Usually Proved by the Method of Limits.—Dr. H. Y. Benedict, University of Texas.
6. A Course of Study in Mathematics for High Schools.—C. O. Calloway, High School, Waco.
7. Discussion of Papers.
8. Business Meeting.

Prof. Dodd's paper was read by Mr. T. H. Rogers of Houston. The papers were carefully prepared and were fully and profitably discussed by many members of the section for an hour or more.

Prof. C. O. Calloway of Waco was elected chairman for the ensuing year. The meeting then adjourned to meet in 1916.

Two of the papers read are printed in this issue of the Bulletin.

PROOFS BY APPROXIMATION METHODS OF CERTAIN
THEOREMS OF GEOMETRY USUALLY PROVED
BY THE METHOD OF LIMITS

(Read before the Mathematics Section of Texas State Teachers'
Association, November 26, 1915.)

The phrase "usually proved by the method of limits" is more polite than true. The proofs of the theorems that involve limits and incommensurables are so defective as presented in most of our textbooks that they are no proofs at all. Pedagogically such proofs are worse than useless and many of our better and clearer headed teachers are omitting them. Logically each of these proofs has at least one flaw in it. The mathematical objections to these so-called proofs having been summarized in the University of Texas bulletin on "The Teaching of Plane Geometry,"* and more fully treated by E. R. Hedrick, "Approximations and Approximate Processes," in the November and December, 1908, numbers of School Science and Mathematics, it is not necessary to repeat them. All that needs to be attempted here is to indicate the proofs by approximation methods of two or three of the limit-incommensurable group of theorems. The proofs of the other theorems in the group are entirely similar.

In geometry the notions concerning measurement and units that the student has gained in pregeometric days should be made somewhat more precise. In particular, emphasis should be placed on the fact that measurement consists in dividing, with or without a remainder, a geometric figure up into portions each of which *fits* the unit. From this it follows that the unit must be of the *same kind* as the figure measured. Thus, for example, a unit of volume must be a solid, and a piece of a straight line is not a suitable unit in measuring an arc of a circle. Length being defined in terms of a piece of a straight line, a circle, strictly speaking, has no length. A circle *fits* no part of a straight line and a straight line fits no part of a circle.

*See supplement to this article.

That the measure of a quantity is a number, that the smaller the unit of measurement the larger is this number, that in general the remainder becomes smaller when the unit is chosen smaller, are all facts quite easy for the pupil to understand. Nor does he have any difficulty in realizing clearly that practically the unit cannot be made so small as to escape our senses and that, as a consequence, all practical measurements are approximations only.

The ratio between two figures of the same kind having been defined as usual, as the numerical quotient that arises from dividing the numerical measure of one by the numerical measure of the other, informal approximation proofs may be given.

Theorem. In the same circle or equal circles, the ratio between the numerical measures of two central angles is equal to the ratio between the numerical values of the intercepted arcs.

Proof. Choose some submultiple u , of one of the arcs U for a unit of arc and take the angle a , that belongs to this unit arc as the unit angle.

Then

$$pu=U \text{ and } pa=A$$

where A is the angle that intercepts the arc U and p is the numerical measure of U and of A .

Let the unit arc u be superposable q times on the second arc V , either with or without a remainder. Then the unit angle A is superposable q times on the second angle B , which intercepts the arc V , either with or without a remainder. Since the remainder when it exists is less than u in one case and a in the other we have

$$qu \leq V < qu + u, \quad qa \leq B < qa + a$$

the equality sign applying when there is no remainder.

Since

$$pu=U, \text{ and } pa=A$$

division gives

$$\frac{q}{p} \leq \frac{V}{U} < \frac{q+1}{p}, \quad \frac{q}{p} \leq \frac{B}{A} < \frac{q+1}{p}.$$

Hence each of the ratios that arise from comparing

V with U and B with A is *between* $\frac{q}{p}$ and $\frac{q+1}{p}$. Hence

each ratio is nearer q/p than is $\frac{q+1}{p}$. But by taking n and a small enough the integer p may be made as large as we please. Hence $\frac{1}{p}$ may be made as small as we please and consequently $\frac{q+1}{p}$ or $\frac{q}{p} + \frac{1}{p}$ may be made as close as we please to q/p . Hence V/U and B/A are each as near as we please to q/p and therefore as near as we please to each other. They are therefore equal to any degree of approximation needed.

Theorem. The area (numerical) of a circle is half the product of the radius (numerical) by the length (numerical) of the circle.

Assumptions. (1) The length of a circle is greater than the perimeter of an inscribed polygon and less than that of a circumscribed polygon.

(2) The area of a circle is greater than the area of the inscribed polygon and less than that of the circumscribed polygon.

Proof. Circumscribe and inscribe regular polygons of n sides. Let C and I be the perimeters of the circumscribed and inscribed polygons. Let p be the apothem of the inscribed polygon. The radius of the circle, r , is the apothem of the circumscribed polygon. Let A be the area of the circle and L its length. Obviously, by the assumptions and the known areas of the regular polygons

$$\frac{1}{2} p I < \frac{1}{2} p L < \frac{1}{2} r L < \frac{1}{2} r C$$

$$\frac{1}{2} p I < A < \frac{1}{2} r C$$

where $\frac{1}{2} p I$ and $\frac{1}{2} r C$ are the areas of the inscribed and circumscribed polygons respectively.

By taking n large enough p can be made as close as we please to r , I to C , $p I$ to $r C$. Hence $\frac{1}{2} r L$ and A can each be made as close as we please to $r C$ and therefore as close as we please to each other.

Hence the area of a circle is given to any approximation needed by the formula $\frac{1}{2} r L$.

Theorem. The area (numerical) of a rectangle is equal to the product of the lengths (numerical) of its sides.

Dealing with a rectangle of sides a and b , where the unit of length is u , we can write

$$\begin{aligned} pu &\leq a(p+1)u \\ qu &\leq b(q+1)u \\ pq[u^2] &\leq ab(pq+p+q+1)[u^2] \end{aligned}$$

S being the numerical measure of the area of the rectangle, with sides $(p+1)u$ and $(q+1)u$, I being that of the inside rectangle with the sides pu and qu

$$S-I < (p+q+1)[u^2] = (pu+qu+u)u$$

Since pu cannot exceed a and qu cannot exceed b , $(pu+qu+u)u \leq (a+b+u)u$. Obviously $(a+b+u)$ can be made as small as we please by taking u small enough. Hence pq is the numerical measure of the area of the rectangle in terms of the unit square, to any degree of approximation desired.

Proofs of the other theorems to which approximation methods apply are entirely similar. The general course of such proofs is as follows: (a) Two numbers A and B are to be shown to be equal to any degree of approximation. (b) Each is shown to be greater than some third number, say I . (c) Each is shown to be less than some fourth number, say S . (d) It is shown that a sufficiently small unit of measurement u , causes S and I to be as close as we please to each other. (e) Hence A and B are as close as we please to each other. (f) Hence A equals B to any degree of approximation.

SUPPLEMENT

For the benefit of those who may not have seen the "Bulletin on the Teaching of Plane Geometry" referred to in the foregoing paper, that part of it relating to Incommensurables and Limits is reprinted here. The complete bulletin may be had by directing a request to the Visitor of Schools, University of Texas, Austin, Texas.

Limits; Incommensurables**A. Approximations**

Some harm has come to the common sense element in school mathematics from the placing of an unwise emphasis on a somewhat mysterious "mathematical accuracy."

As a matter of fact, a large number of interesting and important mathematical processes are essentially processes of approximation. Moreover, most physical measurements are mere approximations to the true measures; however accurate the measurement, there is always some residual error.

Observational Approximations

In observational approximations the error cannot be made smaller and smaller if we happen to desire greater and greater accuracy. Beyond a certain accuracy physical measurements are not able to go. In rough measurements the error is relatively large, in accurate measurements it is relatively small.

Our mathematical textbooks, influenced by practical considerations and the needs of students making measurements in shops and laboratories, are beginning to pay more attention to approximate calculations and to the effect of errors in data on the accuracy of the result. As a consequence a better arithmetical experience is arising. No longer are students asked to figure out the exact number of ounces in a pile of coal or the number of cubic inches in the Pacific Ocean. "Nearly," "just

about," "on the average" are phrases that should be heard oftener in the school room.

The waste of time involved in making calculations more accurately than the data justify should be fully appreciated by the pupils, who should not calculate the area of a rectangular pine board which is 14.37689 inches long by 3.67492 inches broad, nor use π to six decimal places in calculating the area of a circle whose diameter has been obtained by stepping it off.

Students, moreover, for very practical purposes, should learn to estimate the accuracy of results which are based on measurements subject to a certain percentage of error. For instance, how accurately is the volume of a cube known when it is calculated by cubing the measured length of an edge, the measurement being subject to an error of about one part in a thousand?

Again, to compare observational approximations with mathematical approximations, let him, for example, get the square root of 4.73 by drawing a semi-circle of diameter 5.73 and measuring the mean proportional to 1 and 4.73 drawn perpendicular to the diameter at the point 4.73 from one end. Let him also get the square root of 4.73 in the usual way. He will easily discover that the first, the observational approximation process, cannot compete in accuracy with the second, the mathematical approximation process.

It is important that approximation processes be recognized as such and not confused with exact processes. For example, when we set out to express the number whose square is 2 as a decimal fraction we are of necessity using an approximation process because there is no decimal (or common) fraction whose square is *exactly* 2.

It is to be noticed that in the domain of approximate calculations and constructions it is nearly as easy to trisect an angle as to bisect it, nearly as easy to inscribe a regular heptagon in a circle as to inscribe a regular pentagon, use being made in each case of only an unmarked straight edge and compasses. With such equipment the trisection of the angle and the inscription of the heptagon are impossible only in the ideal, the exact sense. In actual operations the approximate construction is the useful one, the exact construction possessing mainly ideal significance.

Mathematical Approximations

In mathematical approximations the error can be made smaller and smaller toward zero if we happen to desire greater and greater accuracy.

Two elements enter into a mathematical approximation process:

- (1) The number to be approximated is perfectly defined.
- (2) There must be some way of calculating a number which differs from the defined number by less than any small number, other than zero, *chosen in advance*.

Usually the process of mathematical approximation is a process of *successive* approximations. Having gotten a number close to the defined number it is possible to use this number to obtain a number yet closer.

Further, mathematical approximation usually involves what is called "control" of the error. By this is meant that it is possible, without determining the actual error, to prove that it cannot exceed a certain amount.

If some method exists by which the defined number may be calculated exactly, such a method is, of course, not approximation process. Even when an exact method of calculating the desired number exists an easier method of approximate calculation may be used *if it gives a result accurate enough for the matter in hand*.

As an illustrative example, let us consider x in the equation $10^x = 7$. Is there an x that satisfies this equation?

Obviously no integral power of 10 is equal to 7. It is almost as clear that no fractional power of 10 is equal to 7. For p and q being integers, $10^{p/q} = 7$ necessitates $10^p = 7^q$, which is impossible since 10^p is an even integer and 7^q is an odd integer. No integral or fractional x exists that satisfies this equation. We can say that no x satisfies this equation or we can *define* x so as to satisfy this equation. We choose the last alternative.

Definition of x . If we choose p and q so that $10^p < 7^q$ then $x > p/q$; if $10^p > 7^q$ then $x < p/q$.

Approximation Process for Calculating a number as close to x as we please.—Let n be a number such that $n - x$ or $x - n$ is less than s where s is a number chosen in advance as small as

we please, but not zero. An integer q can always be chosen larger than $1/s$, so that $1/q$ is smaller than s . 7^q may now be found by a process of multiplication. The number of digits in 7^q can be found by counting them. Suppose there are p digits in 7^q . Then 10^{p-1} is less than 7^q and 10^p is greater than 7^q . Hence by definition,

$$\left\{ \frac{p-1}{q} \right\} = \frac{p}{q} - \frac{1}{q} < x < \frac{p}{q}.$$

Control of approximation. Clearly $x - \left\{ \frac{p}{q} - \frac{1}{q} \right\}$ is less than

$1/q$ and so also is $\frac{p}{q} - x$. Let n be either p/q or $\frac{p-1}{q}$ we have $n - x$ or $x - n$ less than s where s is a certain number chosen in advance and as small as we please but not zero.

B. Lengths, Areas, Volumes.

In the light of what has just been said concerning approximations and of the algebraic illustration just given, we can now discuss satisfactorily such geometric terms as length, area, volume.

Shall such terms be defined or are they so fundamental as to stand in no need of definition? For brevity let us confine our attention to areas; what will be said of areas will be found to apply to lengths and volumes.

What is meant by a square unit of area is so simple that the geometric experience of the pupil may be expected to supply the meaning. Similarly, without definition, we may expect a knowledge of what is meant by the area of a rectangle whose length and breadth are each measured by an integral number of units. But what about the area of a triangle or circle or closed curve that cannot be completely cut up into unit squares?

Just as in our previous algebraic example (1) we may say that the triangle or circle or closed curve has no area, or (2) we may define what we mean by area in such cases. The latter alternative is the one that is chosen.

Definition of Area. Suppose the plane containing the closed curve to be covered in rectangular array with unit squares, so

that all of the interior of the curve and part of the exterior is covered with squares. The area of the curve is defined to be greater than the total number of all the unit squares that fall wholly inside the curve but less than the total number of unit squares that fall wholly or partly within the curve.

Approximation Process: Lay down the squares as described. Count those that fall wholly within the curve: suppose there are p of them. Count those that fall partly within and partly without the curve. Suppose there are q of them. If a is the area of the curve, then, in terms of unit squares, u ,

$$pu < a < (p+q)u.$$

Successive Approximations: Take for the unit square a smaller and smaller square. This involves more laying down of squares and more counting to get p and q .

Control: Prove that qu is small compared to pu or $(p+q)u$ when the unit square is small.

This definition of area is well worth giving. It should be accompanied by illustrative figures from which the student can see that, as the unit square is taken smaller and smaller, (1) p gets larger and larger, (2) q gets larger and larger, (3) q gets smaller and smaller *compared with* p , (4) pu increases, (5) $(p+q)u$ decreases, (6) qu gets smaller and smaller toward zero.

That this approximation applies to any closed curve, polygon, etc., including squares is obvious. It affords a means of comparing any two areas such as a square and a circle, or two similar triangles, etc. For, a and A being two areas, we have

$$pu < a < (p+q)u, \quad Pu < A < (P+Q)u$$

whence

$$\frac{Pu}{(p+q)u} < \frac{A}{a} < \frac{(P+Q)u}{pu}$$

To show that $\left| \frac{P+Q}{p} - \frac{P}{p+q} \right|$ is small when q/p and Q/P

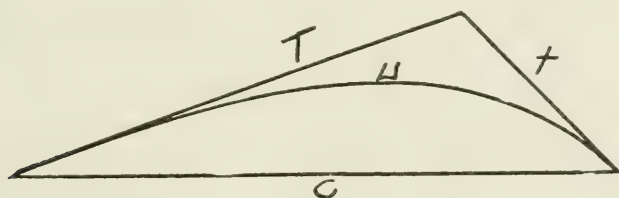
are small now furnishes an excellent problem for the student.

If attention were directed only to areas which can be measured by laying down a unit square a certain number of times the term area would need no definition. But the interior of a triangle or circle can *never be wholly covered by squares lying entirely*

within the figure and a rectangle cannot *always* be wholly filled with squares.



The definition and comparison of *volumes* differs in no important respect from the definition and comparison of areas. In defining the length of a curve, however, a modification is necessary, but the fundamental ideas remain the same. The length of an arc which is wholly convex or concave is, *by definition*, greater than the length of its chord and less than the sum of the segments of the tangents at its extremities reaching from the points of tangency to the intersection of the tangents. In the figure



$$c < u < t + T.$$

A variety of problems now arises. For example, by drawing a circle on co-ordinate paper, a student can determine p and q by actual count and get the area of a circle approximately. He can find, to a certain degree of accuracy, that the areas of circles and other similar figures are to each other as the squares of their corresponding parts. He can find in a very natural and easy way that the perimeters of two figures can get nearer and nearer to coincidence with each other *without* the *lengths* of the perimeters getting nearer together. Thus, as the unit square is taken smaller and smaller he can see that the *primeter* of the figure formed by all the squares wholly within a circle is *approaching* the circle but the *length of this perimeter is approaching, not the length of the circle, but the length of four times the diameter of the circle.*

In elementary mathematics, lengths and areas and volumes should be defined and compared by the above approximation process. The ideas involved are simple and helpful, and the process is not beset with logical difficulties and fine distinctions. Use is made of clear, concrete, practical arithmetic, and the results are ample for practical purposes.

C. *Limits and Incommensurables*

That limits should not be mentioned in an elementary geometry is a corollary of the last paragraph. This apparently radical statement may be further defended by additional discussion.

The treatment of limits, the so-called "limit proofs" and "incommensurable cases" given in our elementary geometries are (1) harmful to the pupil and (2) logically defective.

To begin with, a limit is usually defined incorrectly, the definition being vitiated by the statement that "a variable cannot reach its limit."

As a matter of fact, some variables reach their limits, others cannot. Thus as x is chosen nearer and nearer to 2, x^2 gets closer and closer to 4 as a limit and nothing prevents x^2 from reaching 4. Again, a secant is a straight line which cuts a curve in two distinct points. As one of the points is made to move along the curve nearer and nearer to the other, the secant gets nearer and nearer to the tangent as a limit. Nothing prevents the moving point from reaching the fixed point and even passing through it. On the contrary, as x is chosen larger and larger

$\frac{x+2}{x+1}$ gets nearer to 1 as a limit but never reaches it. There is

no x that satisfies the equation $\frac{x+2}{x+1} = 1$. Again, as the number of

sides of an inscribed polygon is made larger and larger, the area of the polygon approaches the area of the circle as a limit, but no polygon is a circle.

Involved in all the proofs of incommensurable cases in most of our texts is the following fallacy. The books give as a basis for their proofs "if two variables are always equal to each other their limits are equal" or "if two variables are always equal and each approaches a limit their limits are equal." These state-

ments are equally obvious and equally useless. In their proofs, however, they cannot use either of the statements just quoted, and they use without explicit formulation "if two variables depend on a third variable, and if they are equal for all values of the third variable except one, then they are equal for that

value." Thus $\frac{x^2-4}{x-2}$ and $x+2$ are two variables that depend on x and they are equal for all the values of x except 2. They are not equal when x is 2, one being meaningless and the other 4.

Based on an unexpressed statement which is not always true, the proofs as given in most texts, are logically defective. Further, strictly logical proofs are in these cases difficult, a fact made plain by the failure of authors to give them.

Nor are such logical proofs, when attained, of such a character as to make them useful to young pupils. Therefore, only processes of mathematical approximation should be used.

Incommensurables involve limits and like limits should not be stressed in elementary work. The texts often talk about them without even proving that they exist. If any theorem at all concerning incommensurables is established it should be their existence theorem (unless its proof is too complicated for students of this grade to comprehend) for their existence is by no means self-evident.

Having established the existence of incommensurables by proving that at least one exists, having pointed out the significance of the existence of such numbers, the subject of incommensurables may well be dropped, all proofs of theorems for incommensurable cases being dropped along with them.

Now it is an easy matter to show that incommensurables exist. That no fraction exists whose square is 2 is rather obvious. For suppose p^2/q^2 is 2, p and q being positive integers. Then $p^2=2q^2$. Now the square of an even integer is even and the square of an odd integer is odd. $2q^2$ is even whether q is even or odd, therefore, $p=2s$. Then $p^2=(2s)^2=4s^2=2q^2$, or $q^2=2s^2$. Whether s is even or odd $2s^2$ is even, q^2 is even, and therefore q is even. Hence the fraction whose square is 2 must have an even numerator and an even denominator. But this fraction may be put in its lowest terms when it cannot have both its numerator and its denominator even. No fraction can meet the

contradictory requirements involved in the hypothesis that its square is 2. Hence the square root of 2 is not a fraction.

We can at this point say (1) that 2 has no square root, or (2) that it has a square root. If we choose (2) incommensurables exist; if we choose (1) the hypotenuse of a right triangle whose sides are of length 1 has no length. This awkward result leads us to assume that a number exists whose square is 2.

In elementary work the existence of the incommensurable may be left unmentioned and unproved without damage. Surely, however, incommensurables should not be mentioned and proofs concerning them inserted, unless the existence of at least a few incommensurables is also established. Before discoursing on certain details concerning the ear of an elephant it is well to make sure that at least one elephant ear exists or has existed. That incommensurables exist is by no means obvious, their existence being a matter of definition. To prove that two numbers are incommensurable is often difficult. For example, until 1882 mathematicians had not succeeded in proving that the ratio of the length of a circle to the length of its diameter was *not equal* to the ratio of any integer to any other integer. The fine distinctions that occur in dealing with incommensurables show us that they should be dealt with very sparingly, if at all, in high school work.

LITERAL ARITHMETIC

Students taking up the study of elementary algebra for the first time meet a number of new and more general notions not found in their previous experience. These new notions are crowded in upon the mind of the student in such a manner that before he has comprehended clearly one of them he is forced to take up another. The transition from the arithmetic of definite numbers to the more general statements of algebra is often too rapid for the average student. In most of our schools no preparation is made for algebra until the student is suddenly confronted with the subject in the first year of the high school. In a time entirely too limited, he is asked to make intelligible statements about new concepts which are still vague and indefinite to him. Many failures are found along the way and many others stumble over difficulties that should not be encountered. The fault is with the method of procedure rather than with the student. A preparation should begin in the grades below the high school and the transition from arithmetic to algebra be made slowly and intelligently.

Among some of the difficulties found by students first beginning the study of algebra may be mentioned:

(a) The representation of numbers by means of letters. This is a step from the particular to the more general notion of number. It takes some time to train pupils to think in this more general language of numbers.

(b) The use of letters together with the symbols of operation to indicate certain arithmetical processes.

(c) The introduction and interpretation of the negative number. It is with the use of this kind of number that algebra may be said to begin.

(d) The introduction of fractional and negative indices.

(e) The notion of the variable, the graph and the function.

With very little training in (a) and (b) the average text in elementary algebra introduces the student to a consideration of (c), a notion of number much more difficult than (a) and (b). Without an intelligent background, the pupil begins with nega-

tive number and soon falls into a mechanical way of handling signs and symbols and hence loses the logical processes so characteristic of algebra. In other words algebra becomes to the student a piece of mechanism for turning out certain results. Hence, the method of taking up one difficulty at a time applies to algebra probably more than to any subject in the schools. To be able to think mathematically in the use of the more general representations of number is not attained in a day or a week. Such a power is the result of a mental growth and a mental growth is a matter of time and training.

There is no hard and fast boundary line between arithmetic and algebra, but for the sake of clearness I shall designate in what follows as falling under the head of arithmetic all processes of addition, subtraction, multiplication and division, by the use of letters and symbols of operation, that do not involve the use of negative number. Some writers have called this *generalized arithmetic*, but in what follows it will be designated as *literal arithmetic*.

In some of the more recent texts on arithmetic the authors have recognized the value of general statements of rules and formulas by the use of letters and signs and have added a chapter giving some of the elementary processes and expressions. As a rule, these chapters have been of little value, first because too little space and matter have been given to develop the proper mental growth and power and second because they are usually added as an appendix when as a matter of fact the training and methods developed should be used in the last study in the grades of such arithmetical applications as percentage, interest, proportion, mensuration, etc. The expression of the rules of these applications by the use of letters and formulas will be retained much more tenaciously by the mind than by long verbal statements. Formulas may be said to be a translation of verbal rules and statements into the literal notation. On account of the abbreviation in oral or written form they are also said to be "short hand" expressions of verbal statements.

Literal arithmetic has therefore two claims for a place in the grades preceding the high school; first, in giving greater power in handling applications in arithmetic, second in making a preparation necessary for beginning algebra. The writer knows no

text that will give an adequate training in literal arithmetic. The teacher looking for suggestions may find help in some introductory chapters of elementary algebra.* By a skilful use of such hints he may build up a series of exercises that could be used in such work. A regular time on the schedule should be given for several months using a well graded series of exercises. I would suggest one or two lessons a week taken from the time usually given to arithmetic. In what follows I shall suggest an outline beginning with the most elementary notions leading up to the solution of simple equations of one unknown and to the translation of verbal statements into formulas.

I. Expressions involving only multiples of the letters. These exercises should be first oral and then written.

Examples:

What is the double of a ? $2a$? $3a$?

If a is an integer is $2a$ even or odd?

If a is an odd number write three odd numbers.

The pupil may be inducted into the use of letters as follows:

Examples:

{ How many dimes in 9 dollars?

{ How many dimes in k dollars?

{ A man with 5 oranges bought 6 more and had $(5+6)$ oranges.

{ A man with 5 oranges bought b more and had $(5+b)$ oranges.

{ A man with a oranges bought b more and had $(a+b)$ oranges.

At this place the solution of simple equations may begin. Thus in $x+3=11$ the pupil will see that $x=8$ from his knowledge of addition.

The difficulty of these may be increased by giving such as

$$2x+5=13$$

$$3x-4=14$$

the equations being selected so that x is not a negative number.

After the solution of equations are given, simple problems may be given for statement. For example:

(a) Find two consecutive integers whose sum is 27.

(b) James and John have 31 marbles and John has 7 less than James, how many has each?

Practice in substitution should begin in these elementary no-

*See the Wentworth-Smith Algebra.

tions. For example give to a , b , c certain values to find the value of such expressions as

$$3a+2b-c$$

where the expressions to be evaluated are so selected that no results indicate negative numbers.

These exercises seem very simple, but it is better to begin in this way with immature pupils than assume too much on their part.

II. Expressions involving products and simple integral powers. Here an extension of the literal notation is made to represent products and powers. Especial attention to be called to the product $ab=a \times b$, a form not permissible with digits. Thus 59 does not mean 5×9 .

Let pupils state in words the meaning of a^2b , $3ab^2c$, etc.

- {If one orange costs 3 cents, then 5 oranges cost 5×3 cents.
- {If one orange costs 3 cents, then a oranges cost $3a$ cents.
- {If one orange costs b cents, then a oranges cost ab cents.

Giving to a , b , c , d certain values find the values of expressions like $5ab^2c$, $7ac^2$, $3a^2b^2c^2$, etc. The practice in substitution is very valuable.

- {If 3 contains x four times what is the value of x ?
- {If a contains x four times what is the value of x ?
- {If a contains x b times what is the value of x ?

- {If one pencil costs 3 cts., how many may be bought for 12 cts.?
- {If one pencil costs 3 cts., how many may be bought for y cts.?
- {If one pencil costs x cts., how many may be bought for y cts.?

Simple problems to be stated may also be given here.

The formula for the area of a rectangle may be found by dividing the rectangle into squares, and areas of particular rectangles may be found by use of the formula.

In giving this very simple outline as a suggestion for elementary training in literal expressions the problems and exercises are more or less artificial. As soon as the pupils are able to understand more difficult statements they may take up formulas and the solution of problems falling in their experience. In the next bulletin I shall indicate how the literal notation may be used in arithmetic.

CONCERNING EXPONENTS

An examination of a great deal of work done in algebra by high school students and college freshmen has convinced the writer that the student's ideas of the nature of exponents, the laws that they obey, and methods of operating with quantities containing them are not only somewhat hazy but in most cases are in a hopeless fog.

Since this is an important topic and one that lends itself to the clearest sort of treatment it seems not out of place to try to set before the teachers of the subject a short discussion that may aid them in an effort to secure clearness and simplicity in presenting the subject to their classes. By a little study and effort the teacher can make this very important subject comprehensible to any student of average intelligence.

Most secondary algebras begin a discussion of the subject of exponents by saying $a \cdot a = a^2$, $a \cdot a \cdot a = a^3$, $a \cdot a \cdot a \cdot a \cdot \dots \cdot a$ (where there are n a 's) $= a^n$ if n is a positive integer. In these equations 2, 3, n are called exponents. Clearly then we have the definition: "An exponent is a number that shows how many times another number has been taken as a factor." (N. B.—It should be explicitly stated that this definition is laid down on the hypothesis that all exponents are positive integers.)

Of this definition the following laws are corollaries:

1. $a^m a^n = a^{m+n}$.
2. $a^m / a^n = a^{m-n}$ if $m > n$.
3. $(a^m)^n = (a^n)^m = a^{mn}$.
4. $(ab)^m = a^m \cdot b^m$.

These are the four usual laws set down for exponents. Since any number is the product of itself and 1, we infer $a^1 = 1a = a$. [Just here it seems proper to protest against allowing students to say that " a^3 means a multiplied by itself three times"—a practice that seems to be well nigh universal. A multiplied by itself three times would give a^2 each time and would never give a^3 .]

If it were the custom to restrict exponents to positive integers there would be no more to say on the subject. The four laws

deduced above are so simple and their interpretation so obvious that no perplexity could arise in a consideration of them.

In regard to the use of numbers other than positive integers as exponents we have at least three courses open to us.

1. Reject them altogether.
2. Give them a different meaning and deduce for them a different set of laws from the ones governing positive integers.
3. Use them and subject them to the same laws and give them whatever interpretation seems most consistent.

The third course is the one followed.

Only negative integers, positive and negative fractions and zero are usually considered in addition to positive integers in elementary texts and this paper will deal only with those. No discussion of the meaning that can be given to irrational numbers used as exponents will be included.

We shall then say

1. $a^m a^n = a^{m+n}$
2. $a^m / a^n = a^{m-n}$
3. $(a^m)^n = (a^n)^m = a^{mn}$
4. $(ab)^m = a^m b^m$

When m and n are any rational numbers. Note that this extension is wholly arbitrary. It is not "deduced," it is not "proved," it is not "inferred." It is a decree that all rational numbers shall, when used as exponents, be subjected to the same laws as if they were positive integers. Note further that no extension of the definition of exponents has yet been made. It is not obvious that any can be made. It may be that no meaning can be assigned to exponents other than positive integers. That is a matter that further consideration must determine.

If we write $10^{5\frac{1}{2}}$, it is clear that $5\frac{1}{2}$ does not show how many times 10 is taken as a factor. One could no more take 10 as a factor $5\frac{1}{2}$ times than he could look out of a window $5\frac{1}{2}$ times. What would 10^{-3} mean? Obviously it does not mean the result of taking 10 as a factor -3 times since that could not have sense any more than to say a boy had gone to school -3 times.

Let us examine the result of applying the laws to fractional exponents. From 1: $a^{\frac{1}{2}}$. $a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^1$. But from our

original definition of an exponent $a^1=a$. $\therefore (a^{1/2})(a^{1/2})=a$. $a^{1/2}$ is, then, one of the equal factors of a . But that is what we call the square root of a . It, therefore, appears that we can interpret $a^{1/2}$ as $\sqrt{a}$. Hence, from 3, $a^{5/12}=(a^{1/2})^{11}=(a^{11})^{1/2}=\sqrt{a^{11}}$, and we say that $10^{5/12}$ means the square root of the eleventh power of 10. In this same way we see that if p and q are positive integers,

$a^q=(a^q)^{1/p}=(a^p)^{q/p}$ and can be interpreted as the q th root of the p th power of a , or $a^q=\sqrt[p]{a^p}$. This gives a meaning to exponents that are positive fractions.

What meaning can be assigned to 10^{-3} ? From 2, $\frac{10^5}{10^8}=10^{5-8}$
 $=10^{-3}$. But $\frac{10^5}{10^8}=\frac{1}{10^3}$. It thus appears that 10^{-3} can be interpreted as the reciprocal of 10^3 . The same interpretation can be deduced for any negative exponent.

It only remains to assign a meaning to a^0 . Evidently the meaning given in the definition of exponents will have no meaning in this case. Turning to the laws let us see how we might get a^0 . From 2, $a^m/a^m=a^{m-m}=a^0$. But it is obvious that $a^m/a^m=1$. Therefore, we can interpret a^0 as 1. This does not *prove* $a^0=1$ as seems to be generally supposed. It merely states that if we extend 2 to all values of m and n that we shall have to say $a^0=1$ in order to preserve consistency. In high school examination papers coming to the notice of the writer no question occurs quite so frequently as "Prove: $a^0=1$."

It would seem that so simple a matter as laying down a definition of positive integral exponents and setting forth the consequent laws governing such exponents could not be a complicated or perplexing matter. When this is clearly comprehended it ought to be easy to explain that the laws (but not the definitions) are extended to exponents that are any rational numbers. The meaning of exponents other than positive integers can then be easily inferred as seen above. In the opinion of the writer, the major part of the difficulty found in this subject arises not from the complexity of the situation nor from the dullness of the pupil, but from a want of clearness and

simplicity in presentation on the part of the algebra writer and the teacher.

Much clearness will be gained and many difficulties of manipulation will disappear if the student is taught to substitute fractional exponents for radical signs in all cases before beginning the usual operations, e.g. Divide

$$\sqrt[4]{\frac{4}{3} \sqrt{a^4}} \div \sqrt[3]{\frac{3}{4} \sqrt{a^4}} \text{ by } 3\sqrt[3]{\frac{3}{a}}$$

This will become simple by passing to fractional exponents for

$$\begin{aligned} \text{then we have } & \frac{2}{3^{1/2}} \frac{a^{3/4}}{3^{1/2}} \div \frac{2}{3^{1/2}} \frac{a^{3/4}}{3^{1/2}} = \frac{4 a^{3/4} - 3 a^{3/4}}{2 (3^{1/2})} = \frac{a^{3/4}}{2 (3^{1/2})} \\ & \frac{3^{1/2}}{a^{1/4}} = \frac{3^{1/2}}{a^{1/4}} = \frac{3^{1/2}}{a^{1/4}} \\ & \frac{a^{3/4} a^{1/2}}{(2) (3^{1/2}) (3^{1/2})} = \frac{a^{5/4}}{18} \end{aligned}$$

This article is not offered as a complete treatment of the subject of exponents, but it is the hope of the writer that it points out the cause of most of the confusion arising in the study of the subject in the secondary schools and indicates how complexity may be replaced by simplicity, haziness by clearness, groping and stumbling by surefootedness.

DOES THE STUDENT GET A FAIR RETURN FOR THE TIME AND EFFORT SPENT ON MATHEMATICS?

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(Read before the Mathematics Section of The Texas State
Teachers' Association, November 26, 1915.)

It is probable that all teachers of mathematics have seen times when they were absolutely sure that the average pupil obtained "Value Received" for the effort expended. After developing work in class and testing by written exercise I have been convinced many times that the effort on the part of the student must have been the indicated even root of a negative number. At least I have been compelled to use my imagination to see much benefit derived. Yet there are many who do exert themselves very strenuously and who do obtain results which are permanent and commensurate with their effort. I have no quarrel with this phase of the subject provided the object of our effort is of some use. Few of us believe as strongly in mental discipline as we once did. It is scarcely necessary for a great percentage of people to delve deeply into mathematics simply for mental discipline at this present day when food for thought can be found in many and varied useful things. I do not wish to be understood as underrating the value of habits of accuracy and logical thinking secured by the study of mathematics, but I do contend that much of the effort that has been expended on mathematics might have been used on things that would have been of great value to the world and probably some of the misfits in life would have been adapted more snugly to their places in society.

A boy comes into the school at the age of six or seven and continues to the age of eighteen. He spends a part of every day, five days in the week, four weeks in the month and eight or nine months in the year on mathematics under various and sundry names. To him algebra has no connection with arith-

metic, geometry is on the other side of the world from algebra, commercial arithmetic is in no way related to anything he has ever had, and trigonometry, if he is so fortunate as to have an opportunity to juggle lines, tangents and logarithms, is altogether foreign to those other things he has generally heard called mathematics. Each in turn is a new country peopled with beings new and ever more strange. At last he reaches his eighteenth milestone and goes out into the world with a roll of parchment under his arm and a vast conceit in his ability to do things and especially in mathematics. What a shock it is to realize that he cannot find the exact cost of a bale of cotton at $12\frac{1}{2}$ cents per pound. He thanks his lucky stars that the price has not soared to $16\frac{2}{3}$ cents. His mother wants the dining room redecorated and she feels sure that now she will beat out that smooth salesman who always sells her two rolls more than she needs, but alas by son's calculation she is short two rolls and that particular pattern is all sold when she goes for more.

He accepts a position in a retail and wholesale establishment, but how he perspires over his $\frac{1}{3}$'s and his $\frac{1}{4}$'s. He actually finds that $\frac{1}{2} \times \frac{1}{2} = \frac{2}{4}$, and that one-half of two-thirds is four-thirds and that there are five halves in ten. Five per cent discount for cash gives him cold shivers, and three, ten and fifteen off is surely Greek or Hottentot or some other barbarous language.

I suppose he has known these things some time in his course, but it is a notorious fact that the high school graduate does not know how to perform simple calculations. Undoubtedly he will be able to pick up the various things he needs easier than if he had never seen them, but the fact remains that he is not able to cope with the situation when it arises.

Who is to blame, the boy, the subject matter, the method or what? Perhaps the boy is to blame occasionally. The teacher is human and may make mistakes. The subject matter considered is not generally what it should be. Cube root and compound proportion have little place in ordinary life. I doubt the wisdom of spending much time on apothecary's weights and measures, Troy weights and anything that is not in general and common use in the community. We ought to spend more time on quick and accurate work in the four fundamentals. We

ought to secure ease in the handling of simple fractions, common and decimal. For some reason fractions have been the bugaboo of all generations. Children seem to have an instinctive fear of them, and yet if we use a little "grace and gumption" we can do much toward smoothing the way of the young explorer in this fearful realm. I am sure we have spent too much time on complex fractions and on numerators and denominators of from two to six places and not enough on mental drill on simple and useful fractions. Probably too much time is spent on rates, and amounts in interest and not enough on bank discount. A large percentage of teachers are themselves in "deep water" in stocks and bonds, how can we expect the children to handle them with ease, and yet this subject is coming more and more into the lives of our people.

True discount, long partial payments, equation of payments, partnership, compound proportion and compound interest are not receiving the attention they once did, and I sincerely hope that many more things under the name of arithmetic will cease to worry us and that we will spend our precious time perfecting the skilful handling of things in general use.

A great many are wasting time spending from three to three and one-half years in algebra and geometry. The elements of algebra can be learned in one year and plane geometry may be digested in the same time. This will be enough to discover adaptabilities and will give training in accuracy and logical thinking. It is a waste of time for the second year pupil in general to try to get a thorough appreciation of the more complex principles and applications of algebra. His mind is not developed enough and the subject should come later in the course.

Holding the student through the course is the greatest problem of the high school. In order to hold him we must resort to various and sundry means of keeping him satisfied or at least partially contented with his duties and his labors. There is a waste of time in cramming down his throat a certain amount of mathematics when he does not like it and has no capabilities for the subject. We often make school such a bugbear by requiring great amounts of things and such scholarly attainments in them that we cause the student to drop out of school. He is hungering for something that he thinks will be useful as well as

satisfying to his soul, and none of us will deny that mathematics is a very poor sustainer of some souls.

I would not detract one jot or tittle from the benefits of the study of mathematics, but I am very strongly averse to stuffing an unwilling mind with something that is unpleasant and thereby taking a very long chance of driving the owner out of school and away from its wholesome influence into the street and various other places of unwholesome influence, and I care not to keep a mind spending much time on mathematics when some other subject will be much more profitable in the vocation which he is going to adopt.

Undoubtedly there is a need of change in our method of handling mathematics in our high schools. Few boys or girls get beyond the idea that it is a meaningless drudgery and a necessary evil. As soon as they pass their examinations they proceed to forget as much as possible and when they enter some vocation where its application is necessary they learn it over and find it very interesting. So they have wasted time that might have been spent profitably and pleasantly.

REMARK ON LOGARITHMS

In approaching the subject of logarithms for the first time, many students find it difficult to hold in mind the principles involved in performing the ordinary arithmetical operations with the new notation. Where such students, as a rule, would make no mistake in the use of exponents, the principles of logarithms are not clearly understood. This is often found to be true even after they have learned to use logarithms correctly in a mechanical way. In order to fix in mind the principles of logarithms and to show their relation to well known laws of exponents, the writer often uses a scheme like the one given below, where the principles of logarithms and the corresponding laws of exponents are written in parallel columns. These parallel statements may be said to be the expression of the same fact in two different languages. Everything given in the parallel columns will be found in the chapter on logarithms in any good text on Trigonometry. The arrangement has been found to be helpful in overcoming some of the difficulties in mastering the principles.

Definition: When we have given

$$a^x=m$$

we define

$$\log_a m=x$$

Exponential Laws.

Logarithmic Principles.

(a) Given $a^x=m$ $a^y=n$. (b) Then by def. $x=\log_a m$, $y=\log_a n$

$$\text{I. } m \times n = a^x \times a^y = a^{x+y} \quad \text{I. } \log_a (m \times n) = x + y = \log_a m + \log_a n \text{ by I and (b)}$$

$$\text{II. } m \div n = a^x \div a^y = a^{x-y} \quad \text{II. } \log_a (m \div n) = x - y = \log_a m - \log_a n \text{ by II and (b)}$$

$$\text{III. } m^k = (a^x)^k = a^{kx} \quad \text{III. } \log_a m^k = kx = k \log_a m$$

$$\text{IV. } \sqrt[k]{m} = (a^{\frac{x}{k}})^{\frac{1}{k}} = a^{\frac{x}{k^2}} \quad \text{IV. } \log_a \sqrt[k]{m} = \frac{x}{k} = \frac{1}{k} \log_a m$$

The teacher should have students to write out these parallel statements repeatedly and see that they are thoroughly understood. By the use of this method much of the mystery and haziness in logarithmic work will be lost, and the ordinary computations in Trigonometry will become more intelligible.

THE STRAIGHT EDGE

Not one student in ten can tell what is meant by saying that a theorem is "proved." That is somebody's fault.

* * * * *

Not one student in ten has a clear idea what is meant by saying an equation is "solved." He thinks that he has the "answer" but does not know that his answer must satisfy the equation. Is anybody to blame for this?

* * * * *

Half the students one meets will define a figure in geometry as "where" or "when" something happens. Have they been indulged or encouraged in this?

* * * * *

Prof. Dryasdust says that he has not had to study his geometry since the second time he took the class over the work and solved all the originals. The professor further says that his classes of late years do not seem to take as much interest in geometry as his former ones. He thinks students are less ambitious than formerly.

* * * * *

Later: Prof. Dryasdust has been succeeded by a young chap just out of college, he says that he knew more mathematics twenty years ago than this youth does now and that experience gets no recognition these days.

* * * * *

Teachers may be "born and not made" but that is a mighty poor excuse for not trying to "make" yourself a better one by studying your subject, by reading what others have found out about it, by studying your students, and most of all by studying yourself to see if you really are as good a teacher as you have fooled yourself into believing.

* * * * *

Maybe you do not get as much pay as you earn. Well, get even with the board by earning twice as much as you get. After you have tried this a while both you and the board may make a discovery.

